University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 11

Reading:
105 Notes 14.1-14.5
Hand & Finch 2.9, 9.7

1.
Discuss the motion of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when the initial conditions are

$$y(x,0) = A\sin\frac{3\pi x}{L}$$
$$\dot{y}(x,0) = 0 \ .$$

Resolve the solution into normal modes.

2.
Discuss the motion of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when (in a certain set of units) the initial conditions are

$$y(x,0) = 4\,\frac{x(L-x)}{L^2}$$
$$\dot{y}(x,0) = 0 \ .$$

Find the characteristic frequencies and calculate the amplitude of the n^{th} mode.

3.
Solve for the motion $y(x,t)$ of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when the initial conditions are

$$y(x,0) = A\sin\frac{\pi x}{L}$$
$$\dot{y}(x,0) = V\sin\frac{5\pi x}{L} \ ,$$

where A and V are constants.

4.
A continuous string (tension τ, mass per unit length μ) is attached to fixed supports *infinitely far away*. At $t = 0$ the string satisfies initial conditions

$$y(x,0) = 0$$
$$\frac{\partial y}{\partial t}(x,0) = \alpha\,\delta(x) \ ,$$

where $\delta(x)$ is a Dirac delta function and α is a constant that can be made arbitrarily infinitesimal, so that the string's slope remains small enough for the usual wave equation to apply. This initial condition is appropriate to the string having been struck at $(x = 0, t = 0)$ with a sharp object.

Compute $y(x,t)$ for $t > 0$.

5.
Show that if ψ and ψ^* are taken as two *independent* field variables, the Lagrangian density

$$\mathcal{L}' = \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi + \frac{\hbar}{2i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)$$

(where $\dot{}$ means $\partial/\partial t$ in this context) leads to the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

and its complex conjugate.

6.
Consider a membrane stretched between *fixed* supports at $x = 0$, $x = L$, $y = 0$, and $y = L$. *Per unit area*, its kinetic and potential energies are

$$T' = \tfrac{1}{2}\sigma\Big(\frac{\partial z}{\partial t}\Big)^2$$
$$U' = \tfrac{1}{2}\beta\Big(\Big(\frac{\partial z}{\partial x}\Big)^2 + \Big(\frac{\partial z}{\partial y}\Big)^2\Big) \ ,$$

where σ is the membrane's mass per unit area, β is a constant that is inversely proportional to its elasticity, and z is its (normal) displacement.

Apply the Euler-Lagrange equations to obtain a partial differential equation for $z(x, y, t)$. Using a trial solution

$$z(x, y, t) = X(x)\,Y(y)\,T(t) \ ,$$

find the angular frequencies of vibration for the *five* lowest-frequency normal modes of oscillation.

7. and **8.** (double problem)
The Lagrangian density (per unit volume) for a charge density $\rho(\mathbf{r}, t)$ and current density $\mathbf{j}(\mathbf{r}, t)$ in the presence of an electromagnetic field $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ is

$$\mathcal{L}' = \frac{E^2 - B^2}{8\pi} - \rho\phi + \frac{1}{c}\mathbf{j} \cdot \mathbf{A} \ .$$

The first term is the Lagrangian density corresponding to the self-energy of the free field, and the latter terms represent the interaction between fields and charges. The self-energy of the individual (point) charges is infinity in classical theory and is omitted. In the above, $\mathbf{A}$ is the *vector potential* defined by

$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

(Gaussian units are used throughout this problem). If you are familiar with relativistic transformations of electromagnetic fields, you may notice that the above Lagrangian density is *Lorentz invariant*, although not manifestly so.

The homogeneous (charge and current independent) Maxwell equations follow directly from the equations relating $\mathbf{E}$ and $\mathbf{B}$ to the potentials. To complete the picture, using ϕ and the three components of $\mathbf{A}$ as four generalized (field) coordinates, apply the Euler-Lagrange equations to $\mathcal{L}'$ to obtain the two inhomogeneous Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$
$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \ .$$